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LETTER TO THE EDITOR

Commuting periodic operators and the periodic Wigner function

J Zak

Department of Physics, Technion–Israel Institute of Technology, Haifa 32000, Israel

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Abstract

Commuting periodic operators (CPO) depending on the coordinate $\hat{x}$ and the momentum $\hat{p}$ operators are defined. The CPO are functions of the two basic commuting operators $\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}\right)$ and $\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}a\right)$, with a being an arbitrary constant. A periodic Wigner function (PWF) $w(x, p)$ is defined and it is shown that it is applicable in a normal expectation value calculation to the CPO, as done in the original Wigner paper. Moreover, this PWF is non-negative everywhere, and it can therefore be interpreted as an actual probability distribution. The PWF $w(x, p)$ is shown to be given as an expectation value of the periodic Dirac delta function in the phase plane.

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1. Introduction

The Wigner function was originally introduced in the early stages of quantum mechanics [1] and has since become of much use both in physics [2] and signal analysis [3]. As is well known, the Wigner function is real, but not everywhere positive, and it might actually take negative values. For this reason it cannot be interpreted as the simultaneous probability distribution for the coordinate x and the momentum p. As was shown in the original paper [1], for a sum of operators $g(\hat{x})$ and $f(\hat{p})$, the following important relation holds:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}p[f(p) + g(x)]W(x, p) = \int_{-\infty}^{\infty} \psi^*(x)\left[f\left(-\mathrm{i}\hbar\frac{\partial}{\partial x}\right) + g(\hat{x})\right]\psi(x)\,\mathrm{d}x, \quad (1)$$

where $W(x, p)$ is the Wigner function

$$W(x, p) = \frac{1}{h}\int_{-\infty}^{\infty} \exp\left(-\frac{\mathrm{i}}{\hbar}pz\right)\psi^*\left(x - \frac{1}{2}z\right)\psi\left(x + \frac{1}{2}z\right)\mathrm{d}z \quad (2)$$

and $\psi(x)$ is the wavefunction of the quantum state. What equation (1) tells us is that for calculating the expectation value of a sum of operators $f(\hat{p}) + g(\hat{x})$, which is given by the right-hand side of equation (1) in quantum mechanics, one can also use the Wigner function in a normal probability calculation (in what follows this will mean that the Wigner function is used as if it were a classical distribution function) as given by the left-hand side of equation (1). Attention is drawn to the fact that $f(\hat{p})$ is a function of the momentum only, and similarly, $g(\hat{x})$ depends only on the coordinate. Wigner's result in equation (1) was generalized by Moyal [4], to include any function $\hat{A}(\hat{x}, \hat{p})$ of the operators $\hat{x}$ and $\hat{p}$. But then for calculating the expectation value of the operator $\hat{A}(\hat{x}, \hat{p})$ one cannot use the normal probability calculation, as suggested by Wigner, and one has to use in equation (1) the classical function $A_{CL}(xp)$ which is obtained from the operator $\hat{A}(\hat{x}, \hat{p})$ according to what is called Weyl's rule [4, 5]. However, in quantum mechanics one encounters operators which are functions, not directly of $\hat{x}$ and $\hat{p}$, but of the following two basic commuting operators:

$$\exp\left(\mathrm{i}\frac{2\pi}{a}\hat{x}\right), \qquad \exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}a\right) \tag{3}$$

where a is an arbitrary constant. Functions of these operators appear in a great variety of problems in physics and signal processing. Examples are the Bloch theory of solids [6], the von Neumann discrete coherent states [7], the magnetic translations [8], the *kq*-representation [9], the quantum Hall effect [10] and different examples in signal processing [11, 12].

2. Periodic Wigner function

In this letter, we construct commuting periodic operators (CPO) based on the operators in equation (3). We show that the Wigner function is, in general, not applicable for calculating expectation values of products of the operators in equation (3) in a normal expectation value calculation, meaning that for them the formula in equation (1) is not applicable. We prove, however, that a previously published function [12, 13] can be defined as a periodic Wigner function (PWF), $w(x, p)$, and that by using $w(x, p)$ one may write the formula in equation (1) for any products of the commuting periodic operators of equation (3), meaning that $w(x, p)$ can be used for the latter operators in a normal expectation value calculation. This PWF $w(x, p)$ is moreover non-negative and can therefore be interpreted as an actual probability distribution for CPO.

We start with the definition of commuting periodic operators $\hat{P}(\hat{x}, \hat{p})$ based on the operators in equation (3) (they are periodic in $\hat{p}$ with the period $\frac{h}{a}$, and in $\hat{x}$ with period a):

$$\hat{P}(\hat{x}, \hat{p}) = \sum_{m,n} P(m, n) \exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right) \exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right), \tag{4}$$

where a is an arbitrary constant, $P(m, n)$ are expansion coefficients and the integers m and n can assume any number of values. We would like to point out that the order of the operators in equation (4) is not important because they commute. An important physical example of operators in equation (4) is the problem of a Bloch electron in a magnetic field [8, 10, 14]. By denoting the expectation value of an operator $\hat{O}$ by $\langle\hat{O}\rangle$, we have the following expression for the expectation value of $\hat{P}(\hat{x}, \hat{p})$ of equation (4) in the state $\psi(x)$ (according to the rules of quantum mechanics):

$$\langle\hat{P}(\hat{x}, \hat{p})\rangle = \int_{-\infty}^{\infty} \psi^*(x) \hat{P}\left(x, -\mathrm{i}\hbar\frac{\partial}{\partial x}\right) \psi(x)\, \mathrm{d}x. \tag{5}$$

On the other hand, if one assumed that the Wigner function can be used for calculating the expectation value of $\hat{P}(\hat{x}, \hat{p})$, as in equation (1), the result would be different. In order to see this we calculate the following integral:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}p \exp\left(\frac{\mathrm{i}}{\hbar}pam\right)\exp\left(\mathrm{i}x\frac{2\pi}{a}n\right)W(x,p)$$
$$= (-1)^{mn}\left\langle \exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)\right\rangle, \tag{6}$$

where on the right-hand side the brackets $\langle\,\rangle$ denote the quantum mechanical expectation value as in equation (5), and on the left-hand side is the normal expectation value calculation, assuming that the Wigner function $W(x,p)$ is suitable for this purpose. Equation (6) shows that for the periodic operator in equation (4), the following result holds:

$$\langle \hat{P}(\hat{x},\hat{p})\rangle = \sum_{m,n} P(m,n)(-1)^{mn}\left\langle \exp\left(\frac{\mathrm{i}}{\hbar}pam\right)\exp\left(\mathrm{i}x\frac{2\pi}{a}m\right)\right\rangle_W, \tag{7}$$

where by the brackets $\langle\,\rangle_W$ we have denoted the left-hand side of equation (6). Equation (7) shows that the Wigner function is not suitable for a normal expectation value calculation of CPO $\hat{P}(\hat{x},\hat{p})$, as defined in equation (4).

We now show how to define a Wigner function which is suitable for calculating expectation values of CPO in the normal expectation value calculation. For this we use the *kq*-representation [9]. In the latter the expectation value of $\hat{P}(\hat{x},\hat{p})$ in equation (4) is (since in the *kq*-representation $P(x,p)$ is a multiplication operator)

$$\langle P(x,p)\rangle = \frac{1}{\hbar}\int_0^a\int_0^{\frac{h}{a}} P(x,p)\left|C\left(\frac{p}{\hbar},x\right)\right|^2 \mathrm{d}x\,\mathrm{d}p, \tag{8}$$

where $C(k,q)\left(k=\frac{p}{\hbar}\right)$ is the *kq*-transform of the wavefunction $\psi(x)$ in equation (1). In [12, 13] the following relation was proven:

$$\frac{1}{2}\sum_{m,n}(-1)^{mn}W\left(x+n\frac{a}{2},\,p+m\frac{h}{2a}\right) = \frac{1}{\hbar}\left|C\left(\frac{p}{\hbar},x\right)\right|^2. \tag{9}$$

Following this relation we introduce the little $w(x,p)$,

$$w(x,p) = \frac{1}{2}\sum_{m,n}(-1)^{mn}W\left(x+n\frac{a}{2},\,p+m\frac{h}{2a}\right) \tag{10}$$

and we call it, by definition, the periodic Wigner function (PWF) for CPO. From equations (8), (9) and (10) it follows that for any operator defined in equation (4) we can use the PWF $w(x,p)$ in the normal probability calculation

$$\langle P(x,p)\rangle = \int_0^a\int_0^{\frac{h}{a}} \mathrm{d}x\,\mathrm{d}p\,P(x,p)w(x,p). \tag{11}$$

Moreover, being equal to the square of the absolute value, $\frac{1}{\hbar}\left|C\left(\frac{p}{\hbar},x\right)\right|^2$ (equations (9) and (10)), of the *kq*-transform, the periodic Wigner function $w(x,p)$ is non-negative everywhere for x and p. It is easy to check that $w(x,p)$ is periodic in x and p with the periods a and $\frac{h}{a}$, respectively,

$$w\left(x+a,\,p+\frac{h}{a}\right) = w(x,p). \tag{12}$$

Also, $w(x,p)$ satisfies the probability requirement

$$\int_0^a\int_0^{\hbar\frac{2\pi}{a}} w(x,p)\,\mathrm{d}x\,\mathrm{d}p = 1. \tag{13}$$

We see therefore that the periodic Wigner function $w(x, p)$ is a *bona fide* distribution function for commuting periodic operators.

Having defined the PWF for CPO in equation (10), it is of interest to consider other distribution functions in this context. A general scheme for constructing distribution functions $D(x, p)$ was considered by Cohen [15]. All his $D(x, p)$-functions satisfy the Wigner equation (1) for the sum of operators $f(\hat{p})$ and $g(x)$. One can, however, check that a special example of a distribution function that satisfies equation (1) for CPO is the Rihaczek function. The latter was first introduced in signal analysis [16]

$$R(x, p) = \frac{1}{\sqrt{h}}\psi(x)F^*(p)\exp\left(-\frac{\mathrm{i}}{\hbar}xp\right), \tag{14}$$

where $F(p)$ is the Fourier transform of $\psi(x)$. In addition to its very interesting property of being applicable in the calculation of expectation values for CPO in the normal probability calculation (a property not possessed by the Wigner function, see equation (7)), the Rihaczek function satisfies also another important relation, which is connected to the *kq*-transform $C(k, q)$ of $\psi(x)$. $|C(k, q)|^2$ has the meaning of the probability of measuring $\frac{p}{\hbar}$ and x modulo $\frac{2\pi}{a}$ and a, respectively. Assuming that the functions $D(x, p)$ in the general Cohen scheme [15] can be given the meaning of a distribution function, the following relation should hold:

$$\sum_{m,n} D\left(x + na, p + m\frac{h}{a}\right) = \frac{1}{\hbar}\left|C\left(\frac{p}{\hbar}, x\right)\right|^2. \tag{15}$$

This relation was called in [13] the quantum postulate on phase space distribution functions. It was proven in [13] that the Rihaczek function $R(x, p)$ satisfies relation (15), while the Wigner function does not. The latter unexpectedly satisfies relation (9). By comparing equations (9), (10) and (15), we can write

$$\begin{aligned} w(x, p) &= \frac{1}{2}\sum_{m,n}(-1)^{mn}W\left(x + n\frac{a}{2}, p + m\frac{h}{2a}\right) \\ &= \sum_{m,n} R\left(x + na, p + m\frac{h}{a}\right) = \frac{1}{\hbar}\left|C\left(\frac{p}{\hbar}, x\right)\right|^2. \end{aligned} \tag{16}$$

This equation gives three different expressions for the PWF. In addition, equations (15) and (16) show that the Rihaczek distribution satisfies the quantum postulate, as formulated in [13]. In this context, it is interesting to point out that, despite the Rihaczek function being complex, the sum of it in equation (16) is real and non-negative.

As was pointed out in the introduction, in order to be able to use Wigner's function in the calculation of expectation values, for an arbitrary operator $\hat{A}(\hat{x}, \hat{p})$, one has to put into correspondence to this operator $\hat{A}(\hat{x}, \hat{p})$ a classical function $A_{CL}(x, p)$ according to Weyl's rule [4, 5, 17]: given an operator $\hat{A}(\hat{x}, \hat{p})$, one writes its integral representation

$$\hat{A}(\hat{x}, \hat{p}) = \frac{1}{\sqrt{h}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\alpha, \beta)\exp\left[\frac{\mathrm{i}}{\hbar}(\alpha\hat{x} + \beta\hat{p})\right]\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{17}$$

The classical function $A_{CL}(x, p)$ corresponding to $\hat{A}(\hat{x}, \hat{p})$ is then given by

$$A_{CL}(x, p) = \frac{1}{\sqrt{h}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\alpha, \beta)\exp\left[\frac{\mathrm{i}}{\hbar}(\alpha x + \beta p)\right]\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{18}$$

By using equations (17) and (18) it is simple to calculate the classical function $\left[\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)\right]_{CL}$ that corresponds to the quantum operator $\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)$ $\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)$ in equation (6). We have

$$\left[\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)\right]_{CL} = (-1)^{mn}\exp\left(\frac{\mathrm{i}}{\hbar}pam\right)\exp\left(\mathrm{i}x\frac{2\pi}{a}n\right). \tag{19}$$

This equation is in full agreement with the result in equation (6), showing that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}p\left[\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)\right]_{CL} W(x,p)$$
$$= \left\langle\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)\right\rangle. \tag{20}$$

The appearance of the factor $(-1)^{mn}$ in equations (6) and (19) is now easy to understand. The basic operators in equations (3), (6) and (19) have a very interesting feature

$$\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}\right) = \exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)$$
$$= (-1)^{mn}\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am + \mathrm{i}\hat{x}\frac{2\pi}{a}n\right). \tag{21}$$

Equation (21) shows that the basic operators in equations (3), (6) and (20) commute, but when adding the exponents the factor $(-1)^{mn}$ appears. This therefore explains the appearance of this factor in equations (6) and (19). There is a sharp distinction between the basic operators in equation (21) for mn = even and the mn = odd. In the first case, for calculating their expectation values, one can use the Wigner function in the normal probability calculation. On the other hand, in the second case, the normal probability calculation by using the Wigner function will give the opposite sign (see equation (6)) to that of the actual result.

3. Examples

We now consider a few examples of using the PWF $w(x, p)$ for calculating expectation values of the periodic operators in equation (21). The first example is for the barrier wavefunction [19]

$$\psi(x) = \begin{cases}\frac{1}{\sqrt{a}}, & |x| < \frac{a}{2}\\ 0, & |x| > \frac{a}{2},\end{cases} \tag{22}$$

where a is an arbitrary constant. The $C(k, q)$-function for this $\psi(x)$ is [9]

$$C(k,q) = \frac{1}{\sqrt{2\pi}}, \quad \text{for} \quad |q| \leqslant \frac{a}{2} \quad \text{and any} \quad k. \tag{23}$$

From equations (9) and (10) it follows that $w(x, p) = \frac{1}{\hbar}$, and therefore for the operators in equation (21) we have

$$\left\langle\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)\right\rangle = \delta_{m0}\delta_{n0}. \tag{24}$$

In this case the expectation values of $\hat{x}$ and $\hat{p}$ are zero.

For the second example we take the coherent state $\psi_\alpha(x)$. For it $|C_\alpha(k,q)|^2$ is [19]

$$\left|C_\alpha\left(\frac{p}{\hbar}, q\right)\right|^2 = \frac{1}{2\pi}\sum_{m,n}(-1)^{mn}\exp\left[-\frac{a^2}{4\lambda^2}m^2 - \frac{\pi^2\lambda^2}{a^2}n^2 + \mathrm{i}\frac{2\pi}{a}(q-\bar{x})n - \mathrm{i}a\left(k - \frac{\bar{p}}{\hbar}\right)m\right], \tag{25}$$

where λ is the width of the Gaussian in the coherent state and

$$\alpha = \frac{1}{\lambda\sqrt{2}}\left(\bar{x} + \frac{\mathrm{i}}{\hbar}\bar{p}\lambda^2\right), \tag{26}$$

$\bar{x}$ and $\bar{p}$ being the expectation values of $\hat{x}$ and $\hat{p}$ in the coherent state. The PWF $w(x, p)$ is (see equations (9), (10) and (25))

$$w_\alpha(x, p) = \frac{1}{\hbar}\left|C_\alpha\left(\frac{p}{\hbar}, x\right)\right|^2. \tag{27}$$

For the expectation values of the operators in equation (21) we have therefore

$$\langle\alpha|\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}as\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}t\right)|\alpha\rangle = (-1)^{st}\exp\left(-\frac{a^2s^2}{4\lambda^2} - \frac{\pi^2\lambda^2}{a^2}t^2 + \frac{\mathrm{i}}{\hbar}\bar{p}as + \mathrm{i}\bar{x}\frac{2\pi}{a}t\right). \tag{28}$$

As was already pointed out, we have here the sign changing result, depending on the product $st =$ even, or odd. What is interesting about the result in equation (28) is that, while $\langle\hat{x}\rangle = \bar{x}$ and $\langle\hat{p}\rangle = \bar{p}$, in the coherent state $\psi_\alpha(x)$, the exponentiated operators have a Gaussian decay.

Finally, we discuss the harmonic oscillator states $\psi_N(x)$. For these states we have for the function $C_N(k, q)$ in the *kq*-representation [20]

$$|C_N(k, q)|^2 = \frac{1}{2\pi}\sum_{m,n}(-1)^{mn}\exp\left(-\frac{a^2}{4\lambda^2}m^2 - \frac{\pi^2\lambda^2}{a^2}n^2 + \mathrm{i}\frac{2\pi}{a}qn - \mathrm{i}akm\right)$$
$$\times L_N\left(\frac{a^2}{2\lambda^2}m^2 + \frac{2\pi^2\lambda^2}{a^2}n^2\right), \tag{29}$$

where $L_N(x)$ is the Laguerre polynomial [21]. Like before, for the coherent states, we find for the harmonic oscillator states

$$\langle N|\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}as\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}t\right)|N\rangle = (-1)^{st}\exp\left(-\frac{a^2s^2}{4\lambda^2} - \frac{\pi^2\lambda^2}{a^2}t^2\right)$$
$$\times L_N\left(\frac{a^2}{2\lambda^2}s^2 + \frac{2\pi^2\lambda^2}{a^2}t^2\right). \tag{30}$$

For $N = 0$, this result coincides with that in equation (28) for $\bar{x} = \bar{p} = 0$.

The appearance of the phase factor $(-1)^{mn}$ throughout the letter is of central importance in dealing with commuting periodic operators (CPO). As was stated before, the periodic Wigner function, as defined in equations (9) and (10), is applicable to the calculation of expectation values of CPO in a normal probability calculation. We draw attention to the definition of CPO in equation (4). Here the operators $\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)$ and $\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)$ appear in a product form. Their order does not matter, because they commute.

Having worked out a few examples of using the PWF $w(x, p)$, let us now connect its definition to the definition of the Wigner function via the expectation value of the phase plane inversion operator. This connection puts $w(x, p)$, as we will see, into a fundamental operator frame. As was shown by Grossmann [17] and Roger [22], the Wigner function $W(x, p)$ can be defined by means of reflections $\hat{\Pi}(x, p)$ about x and p in the phase plane

$$W(x, p) = \frac{2}{h}\langle\psi|\hat{\Pi}(x, p)|\psi\rangle, \tag{31}$$

where $\hat{\Pi}(x, p)$ is the reflection operator about the point (x, p), and $|\psi\rangle$ is the state for which the Wigner function is defined. In equation (31) the inversion operator $\hat{\Pi}(x, p)$ is [17, 22]

$$\hat{\Pi}(x, p) = \frac{1}{2h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\mathrm{d}\beta\exp\left\{\frac{\mathrm{i}}{\hbar}[\alpha(\hat{x} - x) + \beta(\hat{p} - p)]\right\}. \tag{32}$$

Using the definition of the periodic Wigner function $w(x, p)$ in equation (10), and the equations (31) and (32) we have

$$w(x, p) = \frac{1}{h}\langle\psi|\sum_{m,n}(-1)^{mn}\hat{\Pi}\left(x + n\frac{a}{2}, p + m\frac{h}{2a}\right)|\psi\rangle. \quad (33)$$

After some elementary calculation, and by using the well-known formula [23]

$$\sum_s \exp[\mathrm{i}\alpha s] = \sum_s \delta\left(\frac{\alpha}{2\pi} - s\right) \equiv \Delta\left(\frac{\alpha}{2\pi}\right) \quad (34)$$

($\Delta(\frac{\alpha}{2\pi})$ is a notation for the infinite sum preceding it) we find the following result for the sum of the operators in equation (33)

$$\sum_{m,n}(-1)^{mn}\hat{\Pi}\left(x + n\frac{a}{2}, p + m\hbar\frac{\pi}{a}\right) = \sum_{m,n}\exp\left[\mathrm{i}\frac{2\pi}{a}m(\hat{x} - x)\right]\exp\left[\frac{\mathrm{i}}{\hbar}an(\hat{p} - p)\right]$$
$$= h\Delta(\hat{x} - x)\Delta(\hat{p} - p). \quad (35)$$

Here we have used the notation in equation (34) for the infinite sum of the Dirac delta functions. Finally, for the periodic Wigner function $w(x, p)$ in equation (33), we have a very simple and elegant result

$$w(x, p) = \frac{1}{h}\langle\psi|\sum_{m,n}\exp\left[\mathrm{i}\frac{2\pi}{a}m(\hat{x} - x)\right]\exp\left[\frac{\mathrm{i}}{\hbar}an(\hat{p} - p)\right]|\psi\rangle$$
$$= \langle\psi|\Delta(\hat{x} - x)\Delta(\hat{p} - p)|\psi\rangle. \quad (36)$$

Being given by the expectation value of the operator $\Delta(\hat{x} - x)\Delta(\hat{p} - p)$ in the state $\psi(x)$, equation (36) puts the periodic Wigner function into an operator frame.

The expression for the PWF in equation (36) can directly be extended to include what is called the cross Wigner function [3]. Given two states $\psi(x)$ and $\varphi(x)$, the cross periodic Wigner function $w_{\psi,\phi}$ is

$$w_{\psi,\varphi}(x, p) = \langle\psi|\Delta(\hat{x} - x)\Delta(\hat{p} - p)|\varphi\rangle. \quad (37)$$

Respectively, for any of the commuting periodic operators in equation (21), we have

$$\langle\psi|\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}am\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}n\right)|\varphi\rangle$$
$$= \int_0^a\int_0^{\frac{h}{a}} w_{\psi,\varphi}(x, p)\exp\left(\frac{\mathrm{i}}{\hbar}pam\right)\exp\left(\mathrm{i}x\frac{2\pi}{a}n\right)\mathrm{d}x\,\mathrm{d}p. \quad (38)$$

It follows that any quantum mechanical calculation, and not only expectation values, can therefore be calculated for commuting periodic operators by means of the periodic Wigner function $w(x, p)$ and its extension $w_{\psi,\varphi}(x, p)$.

As an example, the calculation in equations (29) and (30) can be extended to include non-diagonal matrix elements between the harmonic oscillator states $|M\rangle$ and $|N\rangle$. We have for the cross periodic Wigner function (equations (27), (37))

$$w_{N,M}(x, p) = \frac{1}{\hbar}C_N^*\left(\frac{p}{\hbar}, x\right)C_M\left(\frac{p}{\hbar}, x\right). \quad (39)$$

Correspondingly, extending the matrix element in equation (30) for two different states, we find for $M \geqslant N$ [24, 25]

$$\begin{aligned}
\langle N|\exp\left(\frac{\mathrm{i}}{\hbar}\hat{p}as\right)\exp\left(\mathrm{i}\hat{x}\frac{2\pi}{a}t\right)|M\rangle \\
&= \int_0^a\int_0^{\frac{h}{a}} w_{N,M}(x,p)\exp\left(\frac{\mathrm{i}}{\hbar}pas\right)\exp\left(\mathrm{i}x\frac{2\pi}{a}t\right)\,\mathrm{d}x\,\mathrm{d}p \\
&= (-1)^{st}\sqrt{\frac{N!}{M!}}\exp\left(-\frac{a^2s^2}{4\lambda^2}-\frac{\pi^2\lambda^2t^2}{a^2}\right)\left[\frac{1}{\lambda\sqrt{2}}\left(sa+\mathrm{i}\frac{2\pi}{a}\lambda^2t\right)\right]^{M-N} \\
&\quad\times L_N^{M-N}\left(\frac{a^2s^2}{2\lambda^2}+\frac{2\pi^2\lambda^2t^2}{a^2}\right),
\end{aligned} \tag{40}$$

where $L_N^{M-N}(x)$ is a Laguerre polynomial [21]. For $M = N$ equation (40) goes over into equation (30).

4. Conclusion

In conclusion, we have defined a periodic Wigner function $w(x, p)$ (PWF) which is shown to be a *bona fide* distribution function for commuting periodic operators. This PWF can be used in a normal probability calculation, as was done in the original Wigner paper [1]. The periodic Wigner function was extended to a cross PWF, and this leads to a general scheme enabling one to perform any quantum mechanical calculation for periodic commuting operators.

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